# On the Investment Network and Development *Online Appendix*

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### **1 Additional robustness**

Comparisons to the investment network estimates in **?**.

**Table 1:** Investment Network Outdegrees: Comparison with vom Lehn and Winberry (2022)



Notes: Comparison between investment network estimates in benchmark years 1972 and 1992. This table reports the row sum of the networks, *outdegrees*.



#### **Table 2:** Investment Network Homophily: Comparison with vom Lehn and Winberry (2022)

Notes: Comparison between investment network estimates in benchmark years 1972 and 1992. This table reports the weight of the diagonal in the network, *homophily*.

#### **Table 3:** Investment Network 41 Sectors: Comparison with vom Lehn and Winberry (2022)

#### **(a)** Outdegrees

#### **(b)** Homophiily



Notes: Comparison between investment network estimates for 41 sectors in VLW in year 2012, corresponding to the year in which we fix the occupational composition of the labor force. This table reports the row sum of the networks, *outdegrees*, as well as the weight of the diagonal in the network, *homophily*.

### **2 Standard Development Accounting**

We run a series of development accounting exercises to benchmark the magnitude of our main finding, that 32.5% of income disparities across countries can be explained by disparities in the investment network.

We follow the benchmark specification in **?** and account for the contribution of measured capital-output ratios, adjusted by capital-output elasticities, i.e.  $\begin{pmatrix} \frac{k}{u} \end{pmatrix}$  $\frac{k}{y}$ )<sup> $\frac{\alpha}{1-\alpha}$ </sup>, as the ratio between its difference to the US and the overall output disparity. So for example, is a country is 4 times poorer than the US and its measured capital output ratio is 2 times smaller, its contribution to output disparities is one half. Figure [1](#page-3-0) reports these contributions in each country in our sample using data from Penn World Tables  $10<sup>1</sup>$  $10<sup>1</sup>$  On average, measured disparities in capital account for 13% of income disparities. We can alternatively compute other two statistics widely available

<span id="page-3-0"></span>

**Figure 1:** Development Accounting

Notes: Levels of income per capita relative to the US in 2017. Share of income disparities accounted for by measured disparities in capital-output ratios,  $\left(\frac{k}{y}\right)^{\frac{\alpha}{1-\alpha}}$  for  $\alpha = 0.3$ .

in the literature. The first one answers the question of what would be the income disparities if capital-output ratios were equalized across countries. This corresponds to the ratio of the variance of income in an economy with no disparities in capital-output ratios and the observed variance of income. This is measure *success1* in **?**. We find that the variance in capital account for 7% of the observed variance in output per worker in our sample.

Finally, we can alternatively compute a slightly different contribution which accounts for the systematic correlation between output-per-worker and capital-output ratios and that is closer to a variance decomposition exercise, **?**. If we measure the ratio of the sum between the variance of capital output ratios and its covariance with output, over the observed variance in output per worker, we find that 17% of the observed disparities can be accounted for by capital.

<span id="page-3-1"></span><sup>&</sup>lt;sup>1</sup>These results are consistent with those obtained using output per hour, although hours is missing for many of the poorest countries in our sample and we therefore benefit output per worker instead.

### **3 Additional Proofs & Derivations**

#### **3.0.1 Closed economy.**

Let the Domar weight of sector *n* be  $\eta_n \equiv \frac{p_n y_n}{p v}$ *pν* , let the share of value added allocated to the production of final goods be  $\zeta_n \equiv \frac{p_n c_n}{p v}$  $\frac{d^2 p}{d p v}$  and the value added share of each sector be  $\tilde{\zeta}_n \equiv$  $\zeta_n + \frac{p_n \chi_n^d}{p \nu}.$ 

<span id="page-4-0"></span>**Proposition .1.** *The equilibrium Domar weights are functions of sectorial investment rate.*

<span id="page-4-1"></span>
$$
\left[I - \Gamma \alpha \Omega - (1 - \Gamma)M\right]^{-1} \zeta \equiv \eta \tag{1}
$$

*or in vector form*

$$
\eta_n = \zeta_n + \sum_{i=1}^N \alpha_i \gamma_i \omega_{ni} \eta_i + \sum_{j=1}^N (1 - \gamma_j) \mu_{nj} \eta_j.
$$

*Proof Proposition [.1.](#page-4-0)* Use the optimality conditions of the firm, to rewrite the expenses in different intermediate and investment goods as a function of gross output, i.e.

$$
\mu_{ni}(1 - \gamma_i) p_{it} y_{it} = p_{nt} m_{nit}
$$

$$
\alpha_i \gamma_i p_{it} y_{it} = r_{it} k_{it}
$$

$$
\omega_{ji} p_{it}^x x_{it} = p_{jt} \chi_{jit}
$$

By no arbitrage, the user cost of capital satisfies,

$$
r_{it} = p_{it-1}^{x} \left[ \frac{1}{R_t} - (1 - \hat{\delta}_i) \frac{p_{it}^{x}}{p_{it-1}^{x}} \right]
$$

where  $1 - \hat{\delta}_i$  corresponds to the adjusted undepreciated value of a unit of capital adjusted along the BGP, i.e.  $1 - \hat{\delta}_i \equiv \frac{1 - \delta_i}{1 + \sigma^i}$  $\frac{1 - o_i}{1 + g_i^k}$ .

Combining the optimality conditions for capital and investment

$$
\alpha_i \gamma_i p_{it} y_{it} = \left[ \frac{1}{R_t} - (1 - \hat{\delta}_i) \frac{p_{it}^x}{p_{it-1}^x} \right] \frac{p_{jt-1} \chi_{jit-1}}{\omega_{ji}} \frac{x_{it}}{x_{it-1}} \frac{k_{it}}{x_{it}},
$$

which we can use to write the feasibility constraint in each sector *n*,

$$
p_{nt}y_{nt} = p_{nt}c_{nt} + \sum_i p_{nt}\chi_{nit} + \sum_j p_{nt}m_{njt}.
$$

We can rewrite this condition as

$$
\zeta_{nt}\frac{y_{nt}}{c_{nt}}=\zeta_{nt}+\sum_{i}\frac{\alpha_{i}\gamma_{i}\omega_{ni}}{\frac{1}{\beta}-(1-\delta_{i})\frac{p_{it+1}^{x}}{p_{it}^{x}}}\frac{x_{it+1}}{k_{it+1}}\frac{x_{it}}{x_{it+1}}\frac{p_{it+1}y_{it+1}}{p_{it}y_{it}}\zeta_{it}\frac{y_{it}}{c_{it}}+\sum_{j}(1-\gamma_{j})\mu_{njt}\zeta_{jt}\frac{y_{jt}}{c_{jt}}
$$

The above define a system of equations across sectors that can be solved for the Domar weights  $\eta_{nt} \equiv \zeta_{nt} \frac{y_{nt}}{c_{nt}}$  $\frac{y_{nt}}{c_{nt}}$ , given investment rates in each sector  $\frac{x_{it+1}}{k_{it+1}}$  and the growth rates of nominal

gross output,  $g_{p_i y_i}$  and investment,  $g_{x_i}$ . Note that the equilibrium Domar weight depends on the full path of output by sector, as well as the investment rates.<sup>[2](#page-5-0)</sup>

That is, the solution to the equilibrium Domar weight depends directly on the path of the growth rates of the Domar weights and the investment growth rates.

$$
\left[I - \frac{1}{\frac{1}{R_t} - \frac{1 - \delta_i}{1 + g^k} p_{it}^x} \Gamma \alpha \Omega \frac{\mathbf{x}_{t+1}}{\mathbf{k}_{t+1}} \frac{g_{\eta_{t+1}}}{g_{\mathbf{x}_{t+1}}} - (1 - \Gamma) M_t\right]^{-1} \zeta_t \equiv \eta_t
$$
 (2)

Along an BGP the Domar weight is a constant and the investment rate is proportional to the discount factor.<sup>[3](#page-5-1)</sup> The adjustment factor along the BGP is then  $\tilde{\beta} = \frac{\frac{1}{\tilde{\beta}}-(1-\hat{\delta}_i)}{\hat{\delta}_i}$  $\frac{\delta_i}{\delta_i}$ .

<span id="page-5-2"></span>**Proposition .2.** *The BGP level of value added in the economy satisfies*

$$
\ln(\nu) = \tilde{\eta}' \Gamma z - \tilde{\eta}' \Gamma(1-\alpha) \ln(\Gamma(1-\alpha)\eta),
$$

*or in vector form*

$$
\ln(\nu) = \sum_{n} \tilde{\eta_n} \gamma_n z_n - \ln(\sum_{n} \gamma_n (1 - \alpha_n) \eta_n) \sum_{n} \tilde{\eta_n} \gamma_n (1 - \alpha_n).
$$

*Proof Proposition [.2.](#page-5-2)* Use the solution and the definition of *ζit* to solve for relative prices, given investment rates.

$$
\frac{p_{it}}{p_{jt}} = \frac{c_{jt}}{c_{it}} \frac{\zeta_{it}}{\zeta_{jt}} = \frac{\eta_{it}}{\eta_{jt}} \frac{y_{jt}}{y_{it}}
$$

These relative prices are useful to define the demand for intermediate inputs, investment and labor, as a function of the vector of sectorial gross output. The demand for intermediate inputs follows  $(1 - \gamma_i) \frac{\eta_{it}}{\eta_{ni}}$  $\frac{\eta_{it}}{\eta_{nt}}y_{nt} = m_{nit}$ , while the demand for investment goods is

$$
\frac{x_{it+1}}{k_{it+1}}\frac{x_{it}}{x_{it+1}}\frac{\omega_{ji}\alpha_i\gamma_i}{\frac{1}{R_t}-(1-\delta_i)\frac{p_{it+1}^x}{p_{it}^x}}\frac{\eta_{it+1}}{\eta_{jt}}y_{jt}=\chi_{jit}.
$$

Total investment in sector *i* defines the level of the stock of capital as

$$
x_{it} = \prod_{j} \left( \frac{x_{it}}{k_{it+1}} \frac{\alpha_i \gamma_i}{\frac{1}{R_t} - (1 - \delta_i) \frac{p_{it+1}^x}{p_{it}^x}} \frac{\eta_{it+1}}{\eta_{jt}} y_{jt} \right)^{\omega_{jt}} \quad \text{or} \quad k_{it+1} = \prod_{j} \left( \frac{\alpha_i \gamma_i}{\frac{1}{R_t} - (1 - \delta_i) \frac{p_{it+1}^x}{p_{it}^x}} \frac{\eta_{it+1}}{\eta_{jt}} y_{jt} \right)^{\omega_{jit}}
$$

.

Along the BGP, the rate of adjustment is  $\tilde{\beta}_i$  as defined before and

$$
x_{it} = \prod_j \left( \frac{\alpha_i \gamma_i}{\tilde{\beta}_i} \frac{\eta_{it+1}}{\eta_{jt}} y_{jt} \right)^{\omega_{ji}} \quad \text{or} \quad k_{it+1} = \prod_j \left( \frac{\alpha_i \gamma_i}{\tilde{\beta}_i \tilde{\delta}_i} \frac{\eta_{it+1}}{\eta_{jt}} y_{jt} \right)^{\omega_{jit}}.
$$

<span id="page-5-0"></span><sup>2</sup>Alternatively, describe this as

$$
\zeta_{nt} \frac{y_{nt}}{c_{nt}} = \zeta_{nt} + \sum_{i} \frac{\alpha_{i} \gamma_{i} \omega_{ni}}{\frac{1}{R_{i}} - (1 - \hat{\delta}_{i}) \frac{p_{n+1}^{x}}{p_{n}^{x}}}\frac{x_{it+1}}{k_{it+1}} \frac{x_{it}}{x_{it+1}} \zeta_{it+1} \frac{y_{it+1}}{c_{it+1}} + \sum_{j} (1 - \gamma_{j}) \mu_{njt} \zeta_{jt} \frac{y_{jt}}{c_{jt}}
$$

Which shows that the equilibrium Domar weights solve a first order equation in differences.

<span id="page-5-1"></span> $3$ Prices of sectorial output move inversely proportional to consumption, which in turn grows at the same rate as final output.

Assume that the supply of labor is inelastic at 1, so the fraction of labor allocated to each sector follows Domar weights adjusted by the sectorial labor expenditure shares in gross output,

$$
l_i^* = \frac{(1 - \alpha_i)\gamma_i p_i y_i}{\sum_i (1 - \alpha_i)\gamma_i p_i y_i} = \frac{(1 - \alpha_i)\gamma_i \eta_i}{\sum_i (1 - \alpha_i)\gamma_i \eta_i}.
$$

For the purpose of describing final demand, it would be useful to define  $\tilde{l}_i=\frac{l_i^*}{\gamma_i(1-\alpha_i)}.$  Also note that the employment allocation is constant along the BGP, because Domar weights are constant.

Final output in each sector is then

$$
y_{nt} = \left[z_{nt}(\prod_i(\frac{\eta_{nt}}{\eta_{it-1}}y_{it-1})^{\omega_{in}})^{\alpha_n}(\tilde{l_{nt}})^{1-\alpha_n}\right]^{\gamma_n}\left[\prod_i(\frac{\eta_{nt}}{\eta_{it}}y_{it})^{\mu_{in}}\right]^{1-\gamma_n}
$$

Taking logs and writing output in matrix form we obtain

$$
\ln(\mathbf{y}_t) = \Gamma \lambda_t + \iota_t + \Gamma \alpha \Omega' \ln(\mathbf{y}_{t-1}) + (1 - \Gamma) M' \ln(\mathbf{y}_t)
$$

where each element of the vector  $\bm\iota$  can be described as  $\iota_{nt}\equiv\gamma_n(1-\alpha_n)\ln(\tilde{I_{nt}})+\gamma_n\alpha_n\sum_i\omega_{in}\ln(\frac{\eta_{nt}}{\eta_{it}})$  $\frac{\eta_{nt}}{\eta_{it}})+$ *γ*<sub>*n*</sub>α<sub>*n*</sub>  $\sum_i$  *ω*<sub>*in*</sub>(ln( $\frac{y_{it-1}}{y_{it}}$  $\frac{j_{it-1}}{y_{it}}$ ) +  $\ln(\frac{\eta_{it}}{\eta_{it-1}})$  $\frac{\eta_{it}}{\eta_{it-1}}) - \ln(\tilde{\beta}_i \delta_i)) + (1 - \gamma_n) \sum_i \mu_{in} \ln(\frac{\eta_{nt}}{\eta_{it}})$  $\frac{\eta_{nt}}{\eta_{it}}$ ).

Notice that the growth rate of gross output adjusted by the growth rate of the Domar weights is nothing else than the growth rate of sectorial prices. Along the steady state of the detrended economy, these are constant and therefore  $\ln(\frac{y_{it-1}}{y_{it}})$  $y_{it}^{i t-1})$  +  $\ln(\frac{\eta_{it}}{\eta_{it-1}})$  $\frac{\eta_{it}}{\eta_{it-1}}$ ) is simply zero.

The solution for gross output is then,

<span id="page-6-1"></span>
$$
\ln(y) = \Xi_t \Gamma z + \Xi_t \iota \tag{3}
$$

where the elasticity of output to sectorial productivity is proportional to  $\Xi_t \equiv (I - \Gamma \alpha_d \Omega' - I)$  $(1 - \Gamma)M$ <sup> $\prime$ </sup>)<sup>-1</sup>. Unlike the Domar weight, these elasticities are not adjusted by the investment rate. Let the price level of the economy be normalized to  $p = 1$ , then aggregate value added is  $\nu = \frac{p_n y_n}{n_n}$  $\frac{n y_n}{n_n}$  for any *n*. We can compute a geometric average of each of the terms using the expenditure shares of consumption and investment  $\tilde{\zeta}_n \equiv \zeta_n + \frac{p_n \sum_i x_m}{\nu}$  $\frac{\sum_i x_{ni}}{v}$  as weights (since these weights add up to 1).

$$
\ln(v) = \sum_{n} \tilde{\zeta}_n \ln(p_n) + \sum_{n} \tilde{\zeta}_n \ln(y_n) - \sum_{n} \tilde{\zeta}_n \ln(\eta_n)
$$

Given a CRS aggregator of sectorial output, the price index for final goods satisfies,  $\ln(p)$  =  $\sum_{n} \zeta_n \ln(p_n)$ . Because final output is the numeraire, the log of the price index equals zero, and therefore the first term in the expression for value added drops up. The weighting of the terms in the sum also include investment shares in value added. Investment shares are proportional to consumption shares in value added whenever sectorial value added shares are proportional to consumption shares across sectors. This is by construction the assumption in canonical models of input-output linkages without capital and we assume that feature here.<sup>[4](#page-6-0)</sup>

<span id="page-6-0"></span><sup>&</sup>lt;sup>4</sup>Alternatively, one can set up the economy so that investment in different capital types is produced through the final good. This economy would also allow us to define the price of value added as a function of sectorial prices in a way that they drop out from the expression above, while allowing for investment shares that need not be proportional to consumption shares. The undesirable feature of this economy is that sector producing for final production and intermediate inputs are decoupled from those producing investment.

We have already characterized the solution to each of the last two terms, in equations [1](#page-4-1) and

[3.](#page-6-1)

$$
\ln(\nu_t) = \tilde{\zeta}_t \mathcal{E}_t(\Gamma \lambda_t + \iota_t) - \tilde{\zeta}_t' \ln(\eta_t).
$$

where we can define the elasticity of value to sectorial TFP as  $\tilde{\eta} \equiv \tilde{\zeta}'$ E.

Unpacking this expression in vector form,  $\tilde{\zeta}_{jt}=\tilde{\eta}_{jt}-\sum_n\gamma_n\alpha_n\omega_{jn}\tilde{\eta}_{nt}-\sum_n(1-\gamma_n)\mu_{jn}\tilde{\eta}_{nt}$ 

$$
\sum_{j} \tilde{\zeta}_{jt} \ln(\eta_{jt}) = \sum_{j} \tilde{\eta}_{jt} \ln(\eta_{jt}) - \sum_{j} \sum_{i} \gamma_n \alpha_n \omega_{ji} \tilde{\eta}_{it} \ln(\eta_{jt}) - \sum_{j} \sum_{i} (1 - \gamma_n) \mu_{ji} \tilde{\eta}_{it} \ln(\eta_{jt})
$$

**Now consider the term,**  $\tilde{\eta}_{t} \iota_{t}$ 

$$
\sum_{n} \tilde{\eta}_{nt} \iota_{nt} = \sum_{n} \tilde{\eta}_{n} \gamma_{n} (1 - \alpha_{n}) \ln(\tilde{l}_{n}) + \tilde{\eta}_{nt} \gamma_{n} \alpha_{n} \sum_{j} \omega_{jn} \ln(\frac{\eta_{nt}}{\eta_{jt}})
$$

$$
+ \tilde{\eta}_{nt} (1 - \gamma_{n}) \sum_{j} \mu_{jn} \ln(\frac{\eta_{nt}}{\eta_{jt}}) - \tilde{\eta}_{nt} \gamma_{n} \alpha_{n} \sum_{i} \omega_{in} \ln(\tilde{\beta}_{i} \hat{\delta}_{i})
$$

which can be rewritten as

$$
\sum_{n} \tilde{\eta}_{n} \iota_{n} = \sum_{n} \tilde{\eta}_{n} \gamma_{n} (1 - \alpha_{n}) \ln(\tilde{\iota}_{n} \iota) + \sum_{n} \tilde{\eta}_{n} \iota(\gamma_{n} \alpha_{n} + 1 - \gamma_{n}) \ln(\eta_{n} \iota)
$$

$$
- \sum_{n} \tilde{\eta}_{n} \iota(\gamma_{n} \alpha_{n} \sum_{j} \omega_{jn} \ln(\eta_{jt}) - \sum_{n} \tilde{\eta}_{n} (1 - \gamma_{n}) \sum_{j} \mu_{jn} \ln(\eta_{j})
$$

$$
- \tilde{\eta}_{n} \iota(\gamma_{n} \alpha_{n} \sum_{i} \omega_{in} \ln(\tilde{\beta}_{i} \delta_{i}).
$$

Therefore the difference in the last two terms of the expression for value added are

$$
\sum_{n} \tilde{\eta}_{n} \iota_{n} - \sum_{n} \tilde{\zeta}_{n} \ln(\eta_{n}) = \sum_{n} \tilde{\eta}_{n} \gamma_{n} (1 - \alpha_{n}) (\ln(\tilde{\iota}_{n}) - \ln(\eta_{n})) - \tilde{\eta}_{n} \gamma_{n} \alpha_{n} \sum_{i} \omega_{in} \ln(\tilde{\beta}_{i} \delta_{i})
$$

We can rewrite the first two terms as a function of influence vectors by replacing the optimal labor demand,

$$
\sum_{n} \tilde{\eta}_{n} \iota_{n} - \sum_{n} \tilde{\zeta}_{n} \ln(\eta_{n}) = \sum_{n} \tilde{\eta}_{n} \gamma_{n} (1 - \alpha_{n}) (\ln(\eta_{n}) - \ln(\eta_{n}) - \ln(\sum_{n} \gamma_{n} (1 - \alpha_{n}) \eta_{n})
$$

Hence,

$$
\sum_{n} \tilde{\eta}_{n} \ell_{n} - \sum_{n} \tilde{\zeta}_{n} \ln(\eta_{n}) = -\ln(\sum_{n} \gamma_{n} (1 - \alpha_{n}) \eta_{n}) \sum_{n} \tilde{\eta}_{n} \gamma_{n} (1 - \alpha_{n})
$$

The equilibrium level of value added in the economy satisfies

$$
\ln(\nu_t) = \tilde{\eta_t}^{\prime} \Gamma \lambda_t - \tilde{\eta_t}^{\prime} \Gamma (1 - \alpha) \ln(\Gamma (1 - \alpha) \eta_t) - \tilde{\eta_t}^{\prime} \Gamma \alpha \Omega^{\prime} \tilde{\beta}^{-1} \hat{\delta}
$$

In vector form

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$$
\ln(\nu_t) = \sum_n \eta_{nt} \gamma_n z_{nt} - \ln(\sum_n \gamma_n (1 - \alpha_n) \eta_{nt}) \sum_n \tilde{\eta}_{nt} \gamma_n (1 - \alpha_n) - \sum_n \tilde{\eta}_{nt} \gamma_n \alpha_n \sum_i \omega_{in} \ln(\tilde{\beta}_i \delta_i)
$$

#### **3.1 Alternative features of the model economy**

#### **3.1.1 Technology choices, general set up.**

We populate the economy by a continuum of firms that produce investment goods for each sector. These firms maximize profits by choosing the amount of investment in each equipment type, but also the intensity of use of each equipment for production following

$$
\max_{\omega_{int}\chi_{int}} r_{nt}x_{nt} - \sum_i p_{it}\chi_{int}
$$

subject to

$$
x_{nt} = \sum_{i=1}^{N} \left( \omega_{int} \chi_{int}^{\sigma_n} \right)^{\frac{1}{\sigma_n}}, \tag{4}
$$

$$
\sum_{i} \xi_{in} \omega_{int}^{\nu_n} = B_n \tag{5}
$$

The production technology is a generalization of the investment aggregator described in equation **??**.

The optimal (interior) choices of firms are characterized by two conditions

<span id="page-8-0"></span>
$$
\left(\frac{\chi_{jnt}}{\chi_{int}}\right)^{1-\sigma_n} = \frac{\omega_{jnt}}{\omega_{int}} \frac{p_{it}}{p_{jt}}\tag{6}
$$

<span id="page-8-1"></span>
$$
\left(\frac{\chi_{int}}{\chi_{jnt}}\right)^{\sigma_n} = \frac{\xi_{in}}{\xi_{jnt}} \left(\frac{\omega_{int}}{\omega_{jnt}}\right)^{\nu_n - 1} \tag{7}
$$

Replacing [6](#page-8-0) into [7](#page-8-1) we obtain

$$
\frac{\chi_{int}}{\chi_{jnt}} = \left(\frac{\xi_{in}}{\xi_{jnt}} \left(\frac{p_{it}}{p_{jt}}\right)^{\nu_n - 1}\right)^{\frac{1}{\sigma_n \nu_n + 1 - \nu_n}}
$$
(8)

as well as

$$
\frac{\omega_{jnt}}{\omega_{int}} = \left(\frac{\xi_{jnt}}{\xi_{in}}\right)^{\frac{1-\sigma_n}{\sigma_n \nu_n + 1 - \nu_n}} \left(\frac{p_{jt}}{p_{it}}\right)^{\frac{\sigma_n}{\sigma_n \nu_n + 1 - \nu_n}}
$$
(9)

Hence, if  $\sigma_n \nu_n - (\nu_n - 1) < 0$  we obtain an interior solution. This is the same as requiring,  $\nu_n$  > 1/(1 –  $\sigma_n$ ). Such a condition requires more curvature in the technology choice than in the investment aggregator. As in ?, if  $\sigma_n < 0$  firms choose to increase the efficiency of the relatively expensive factor, while if  $\sigma_n > 0$ , they increase the efficiency of the relatively cheap factor. At the same time, the relative demand for a particular investment type decreases in its price.

This economy reduces to our benchmark economy as we take the limit when  $\sigma_n \to 0$ . In that case, expenditure shares in the investment aggregators are simply the parameters characterizing the shape of the production possibility frontier in each economy  $\omega_{int} \propto \tilde{\zeta}^{\frac{1}{1-\nu_n}}_{in}$ , and independent of relative prices.

#### **3.1.2 The two sector, two capital example**

We characterize the planner's problem of a two sector, two capital types economy with no labor and no trade. We present the detrended economy and work in continuous time for convenience.

$$
\max_{c_n}\int_{t=0}^{\infty} exp(-\rho t)U(c_1(t), c_2(t))
$$

subject to

$$
\dot{k}_1(t) = x_1(t) - (\delta_1 + g_1^k)k(t)
$$
  

$$
\dot{k}_2(t) = x_2(t) - (\delta_2 + g_2^k)k(t)
$$
  

$$
k_1^{\alpha}(t) = c_1(t) + \sum_{i=1,2} \chi_{1i}(t)
$$
  

$$
k_2^{\alpha}(t) = c_2(t) + \sum_{i=1,2} \chi_{2i}(t)
$$
  

$$
x_1(t) = \prod_{j=1,2} \chi_{j1}(t)^{\omega_{j1}}
$$
  

$$
x_2(t) = \prod_{j=1,2} \chi_{j2}(t)^{\omega_{j2}}
$$

Where  $g_i^k$  is the growth rate of capital along the BGP, which is in turn a combination of the growth rates of technological change of the investment bundle for each sectorial capital.

$$
g_i^k = \sum_{j=1,2} \omega_{ji} g_j^y = \sum_{j=1,2} \omega_{ji} (g_j^z + \alpha_{j} g_j^k),
$$

which can be solved as system of linear equation for the equilibrium growth rates of capital. If the growth rates of sectorial productivity are constant so are the growth rates of capital.

$$
g^k = (1 - \alpha)^{-1} \Omega' g^z
$$

**Optimality** From the principle of optimality, we can solve for sufficient conditions for an optimum.

$$
exp(-\rho t) \frac{\partial U(c_1(t), c_2(t))}{\partial c_i(t)} = \lambda_i
$$

$$
\lambda_i(t) = \mu_j(t) \omega_{ij} \frac{x_j(t)}{\chi_{ij}(t)}
$$
10

$$
\mu_i(t) = -\left(\lambda_i(t)\frac{\partial F(k_i(t))}{\partial k_i(t)} - (\delta_i + g_i^k)\mu_i(t)\right)
$$

We can use the optimality condition for investment to rewrite the dynamics in terms of the dynamics of the co-state *λ*

$$
\frac{\dot{\mu}_i(t)}{\mu_i(t)} = \frac{\dot{\lambda}_j(t)}{\lambda_j(t)} - \frac{x_i(t)}{x_i(t)} + \frac{\chi_{ji}(t)}{\chi_{ji}(t)}
$$

Totally differentiating the optimality condition with respect to consumption, we obtain  $\dot{\lambda}_i(t) = -\sigma \frac{c_i(t)}{c_i(t)} - \rho$ . Hence,

$$
\frac{\dot{\lambda}_j(t)}{\lambda_j(t)} - \frac{x_i(t)}{x_i(t)} + \frac{\chi_{ji}(t)}{\chi_{ji}(t)} = -\frac{\lambda_i(t)}{\mu_i(t)} \frac{\partial F(k_i(t))}{\partial k_i(t)} + (\delta_i + g_i^k)
$$

$$
\sigma \frac{c_j(t)}{c_j(t)} = \frac{\lambda_i(t)}{\mu_i(t)} \frac{\partial F(k_i(t))}{\partial k_i(t)} - \frac{x_i(t)}{x_i(t)} + \frac{\chi_{ji}(t)}{\chi_{ji}(t)} - (\delta_i + g_i^k - \rho)
$$

Which in terms of allocations is simply

$$
\sigma \frac{c_i(t)}{c_i(t)} = \omega_{ii} \frac{x_i(t)}{\chi_{ii}(t)} \frac{\partial F(k_i(t))}{\partial k_i(t)} - \frac{x_i(t)}{x_i(t)} + \frac{\chi_{ji}(t)}{\chi_{ji}(t)} - (\delta_i + g_i^k - \rho)
$$

This shows already why the dynamics of the system will be government by the relative allocation of investment across sectors. The dynamics of sectorial investment can be written in terms of its composition as

$$
\frac{x_i(t)}{x_i(t)} = \sum_{j=1,2} \omega_{ji} \frac{\chi_{ji}(t)}{\chi_{ji}(t)}
$$

From the optimality condition for investment, we know that investment within the sector are inversely proportional to the shadow value of consumption, i.e.

$$
\frac{\chi_{ji}(t)}{\chi_{ji}(t)} = \frac{\chi_{ii}(t)}{\chi_{ii}(t)} + \sigma \frac{c_i(t)}{c_i(t)} - \sigma \frac{c_j(t)}{c_j(t)}.
$$

Therefore, the Euler equation is

$$
\sigma \frac{c_i(t)}{c_i(t)} = \omega_{ii} \frac{x_i(t)}{\chi_{ii}(t)} \frac{\partial F(k_i(t))}{\partial k_i(t)} - (\omega_{ii} - (1 - \omega_{ji})) \frac{\chi_{ii}(t)}{\chi_{ii}(t)} + (1 - \omega_{ji}) \sigma \left(\frac{c_i(t)}{c_i(t)} - \frac{c_j(t)}{c_j(t)}\right) - (\delta_i + g_i^k - \rho)
$$

but because the investment aggregator is constant returns to scale, the second term in the RHS drops out and

$$
\omega_{ji}\sigma_{ci}(t) = \omega_{ii}\frac{x_i(t)}{\chi_{ii}(t)}\frac{\partial F(k_i(t))}{\partial k_i(t)} - (1 - \omega_{ji})\sigma_{ci}(t) - (\delta_i + g_i^k - \rho)
$$

Using an analogous expression for consumption in sector j, we can solve for the Euler equation as a function of primitives

<span id="page-11-3"></span>
$$
\omega_{ji}\sigma\frac{c_i(t)}{c_i(t)} = \omega_{ii}\frac{x_i(t)}{\chi_{ii}(t)}\frac{\partial F(k_i(t))}{\partial k_i(t)} - (\delta_i + g_i^k - \rho) - (1 - \omega_{ji})\left(\frac{\omega_{jj}}{\omega_{ij}}\frac{x_j(t)}{\chi_{jj}(t)}\frac{\partial F(k_j(t))}{\partial k_j(t)} - \frac{\delta_j + g_j^k - \rho}{\omega_{ij}}\right)
$$
(10)

The optimal path is further characterized by

<span id="page-11-1"></span>
$$
\dot{k}_i(t) = \prod_{j=1,2} \chi_{ji}(t)^{\omega_{ji}} - (\delta_i + g_i^k) k_i(t), \qquad (11)
$$

<span id="page-11-0"></span>
$$
F(ki(t)) = ci(t) + \sum_{j=1,2} \chi_{ij}(t).
$$
 (12)

To understand the system dynamics when we need to keep track of consumption, capital stocks, investment paths and the path of relative allocations of investment across sectors. Let *χ<sup>i</sup>* as the total investment coming from sector *i*. Then, we can combine [12](#page-11-0) and [11](#page-11-1) as follows,

$$
\dot{k}_i(t) = \prod_{j=1,2} \chi_j(t)^{\omega_{ji}} \prod_{j=1,2} \kappa_{ji}^{\omega_{ji}} - (\delta_i + g_i^k) k_i(t),
$$

where  $\kappa_{ji} \equiv \frac{\chi_{ji}(t)}{\chi_i(t)}$  $\frac{\partial f(x)}{\partial x_j(t)}$  is the fraction of investment goods produced in sector *j* going to sector *i*.

We can then incorporate the feasibility constraint into the law of motion for capital as follows

$$
\dot{k}_i(t) = \prod_{j=1,2} \left( F(k_j(t)) - c_j(t) \right)^{\omega_{ji}} \prod_{j=1,2} \kappa_{ji}(t)^{\omega_{ji}} - (\delta_i + g_i^k) k_i(t), \tag{13}
$$

Hence, to complete the full dynamics, we need a dynamic equation for  $\kappa_{ji}(t)$  which we obtain from the optimal allocation of investment across sectors. Consider investment goods from sector *j* used in *i* and *i'*, optimality yields,

$$
\frac{\kappa_{ji}(t)}{\kappa_{ji'}(t)} = \frac{\mu_i(t)}{\mu_{i'}(t)} \frac{\omega_{ij}}{\omega_{ji'}} \frac{x_i(t)}{x_{i'}(t)}
$$

Totally differentiating,

$$
\frac{\dot{\kappa}_{ji}(t)}{\kappa_{ji}(t)} - \frac{\dot{\kappa}_{ji'}(t)}{\kappa_{ji'}(t)} = \frac{\dot{\mu}_i(t)}{\mu_i(t)} - \frac{\dot{\mu}_{i'}(t)}{\mu_{i'}(t)} + \frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{x}_{i'}(t)}{x_{i'}(t)},
$$

and now replacing by the euler equation for the dynamic of the shadow price of capital

<span id="page-11-2"></span>
$$
\frac{\dot{\kappa}_{ji}(t)}{\kappa_{ji}(t)} - \frac{\dot{\kappa}_{ji'}(t)}{\kappa_{ji'}(t)} = -(\omega_{ii} \frac{x_i(t)}{\chi_{ii}(t)} \frac{\partial F(k_i(t))}{\partial k_i(t)} - \omega_{i'i'} \frac{x_{i'}(t)}{\chi_{i'i'}(t)} \frac{\partial F(k_{i'}(t))}{\partial k_{i'}(t)}) + (\delta_i + g_i^k) - (\delta_{i'} + g_{i'}^k) + \frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{x}_{i'}(t)}{x_{i'}(t)}
$$
\n(14)

where we can use the definition for investment to write the last two terms as a function of capital, consumption in each sector and the sectorial allocation of investment. Given

$$
x_i(t) = \prod_{j=1,2} (F(k_j(t)) - c_j(t))^{\omega_{ji}} \prod_{j=1,2} \kappa_{ji}(t)^{\omega_{ji}}
$$
  
12

then

$$
\frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{x}_j(t)}{x_j(t)} \approx \omega_i j \frac{\dot{\kappa}_{ji}(t)}{\kappa_{ji}(t)} - \omega_{jj} \frac{\dot{\kappa}_{jj}}{\kappa_{jj}} + \omega_{ii} \frac{\dot{\kappa}_{ii}(t)}{\kappa_{ii}(t)} - \omega_{ij} \frac{\dot{\kappa}_{ij}}{\kappa_{ij}}
$$

By definition  $\kappa_{ji}(t) = 1 - \kappa_{ji}(t)$  and therefore

$$
\frac{\dot{\kappa}_{ji}(t)}{\kappa_{ji}(t)} = -\frac{\kappa_{ji'}(t)}{1 - \kappa_{ji'}(t)} \frac{\dot{\kappa}_{ji'}(t)}{\kappa_{ji'}(t)}
$$
(15)

.

which implies,

$$
\frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{x}_j(t)}{x_j(t)} \approx \left(\omega_{ji} + \omega_{jj} \frac{\kappa_{jj}}{1 - \kappa_{jj}}\right) \frac{\dot{\kappa}_{ji}(t)}{\kappa_{ji}(t)} - \left(\omega_{ij} + \omega_{ii} \frac{\kappa_{ii}}{1 - \kappa_{ii}}\right) \frac{\dot{\kappa}_{ij}}{\kappa_{ij}}
$$

Hence, the dynamics of the allocation of capital [14](#page-11-2)

$$
\frac{\dot{\kappa}_{ji}(t)}{\kappa_{ji}(t)} \left( \frac{1}{1 - \kappa_{ji}} - (\omega_{ji} + \omega_{jj} \frac{\kappa_{jj}}{1 - \kappa_{jj}}) \right) = (\omega_{jj} \frac{x_j(t)}{\chi_{jj}(t)} \frac{\partial F(k_j(t))}{\partial k_j(t)} - \omega_{ii} \frac{x_i(t)}{\chi_{ii}(t)} \frac{\partial F(k_i(t))}{\partial k_i(t)}) + (\delta_i + g_i^k) - (\delta_{i'} + g_{i'}^k) - (\omega_{ij} + \omega_{ii} \frac{\kappa_{ii}}{1 - \kappa_{ii}}) \frac{\kappa_{ij}}{\kappa_{ij}}
$$

An analogous condition for  $\frac{\dot{\kappa}_{ji}(t)}{\kappa_{ii}(t)}$ *κji*(*t*) determines a system of linear equations that can be solved for the dynamics of the investment allocation. Let  $\zeta_{ji} \equiv (\omega_{ji} + \omega_{jj} \frac{\kappa_{jj}}{1-\kappa_{jj}})$  $\frac{\kappa_{jj}}{1-\kappa_{jj}}$ ), then

$$
\frac{\dot{\kappa}_{ji}(t)}{\kappa_{ji}(t)} = \frac{1}{\left(\frac{1}{1-\kappa_{ji}} - \zeta_{ji}\right)\left(\frac{1}{1-\kappa_{ij}} - \zeta_{ij}\right)} \left(\omega_{jj} \frac{x_j(t)}{\chi_{jj}(t)} \frac{\partial F(k_j(t))}{\partial k_j(t)} - \omega_{ii} \frac{x_i(t)}{\chi_{ii}(t)} \frac{\partial F(k_i(t))}{\partial k_i(t)} + \delta_i + g_i^k - \delta_j + g_j^k\right)
$$
\n(16)

Finally,

<span id="page-12-0"></span>
$$
\frac{x_i(t)}{\chi_{ii}(t)} = \prod_{j=1,2} \left( \frac{F(k_j(t)) - c_j(t)}{F(k_i(t)) - c_i(t)} \right)^{\omega_{ji}} \prod_{j=1,2} \frac{\kappa_{ji}(t)}{\kappa_{ii}(t)}^{\omega_{ji}} \tag{17}
$$

**Steady state.** Along the steady state

$$
exp(-\rho t)\frac{\partial U(c_1(t), c_2(t))}{\partial c_i(t)} = \lambda_i
$$

$$
\frac{\lambda_i(t)}{\mu_i(t)} = \frac{\partial F(k_i(t))}{\partial k_i(t)}^{-1} (\delta_i + g_i^k)
$$

and therefore

$$
\frac{x_j(t)}{\chi_{ij}(t)} = \left(\omega_{ij} \frac{\partial F(k_i(t))}{\partial k_i(t)}\right)^{-1} (\delta_i + g_i^k)
$$

The Euler equation and the dynamic condition for the allocation of investment across sectors, implies that in a steady state

$$
\omega_{ii} \frac{x_i(t)}{\chi_{ii}(t)} \frac{\partial F(k_i(t))}{\partial k_i(t)} = \delta_i + g_i^k - \rho.
$$

Note that the one sector neoclassical growth model is a special case of this, where investment is fully specialized in one sector,  $\omega_{ii} = 1$  and  $\frac{x_i(t)}{x_{ii}(t)} = 1$ .

The law of motion for capital, [11](#page-11-1) implies,

$$
x_i(t) = k_i(t)(\delta_i + g_i^k),
$$

replacing back,

<span id="page-13-0"></span>
$$
\omega_{ii} \frac{k_i(t)}{\chi_{ii}(t)} \frac{\partial F(k_i(t))}{\partial k_i(t)} = \hat{\rho}_i.
$$
\n(18)

where  $\hat{\rho}_i \equiv 1 - \frac{\rho}{\delta_{i+1}}$  $\frac{p}{\delta_i+g_i^k}$ .

The optimal allocation of consumption under separable log-utility and using the feasibility constraint in each sector, [12](#page-11-0) satisfies,

<span id="page-13-2"></span>
$$
\lambda_i(t) = \frac{\theta_i}{F(k_i(t))(1 - \frac{\omega_{ii}\epsilon_i}{\hat{\rho}_i}) - \chi_{ij}(t)}
$$
(19)

where  $\epsilon_i \equiv \frac{k_i(t)}{F(k_i(t))}$  $F(k_i(t))$ *∂F*(*ki*(*t*))  $\frac{\partial f(x_i(t))}{\partial k_i(t)}$  is the output elasticity to capital.

Replacing [18](#page-13-0) into the steady state law of motion for capital and using the definition of investment  $x_i$  we obtain

<span id="page-13-1"></span>
$$
\chi_{ji}(t) = \left(k_i(t)(\delta_i + g_i^k)\right)^{\frac{1}{\omega_{ji}}}\left(\omega_{ii}\frac{\epsilon_i}{\hat{\rho}_i}F(k_i(t))\right)^{\frac{-\omega_{ii}}{\omega_{ji}}}
$$
(20)

which we can replace back in the expression for the price  $\lambda_i$ , defining prices as a function of the stock of capital in each sector and parameters.

We can rewrite the steady state [11](#page-11-1) as

$$
\frac{k_i(t)}{\chi_{ii}(t)}(\delta_i+g_i^k)=\left(\frac{\chi_{ji}(t)}{\chi_{ii}(t)}\right)^{\omega_{ji}}
$$

and using the optimal input demands,

$$
\frac{k_i(t)}{\chi_{ii}(t)}(\delta_i+g_i^k)=\left(\frac{\lambda_i}{\lambda_j}\frac{\omega_{ji}}{\omega_{ii}}\right)^{\omega_{ji}}
$$

Replacing back, [18](#page-13-0) we can solve for

<span id="page-13-3"></span>
$$
(\delta_i + g_i^k - \rho) \frac{k_i(t)}{\omega_{ii} \epsilon_i F(k_i)} = \left(\frac{\lambda_i}{\lambda_j} \frac{\omega_{ji}}{\omega_{ii}}\right)^{\omega_{ji}}
$$
(21)

Therefore, equations [20,](#page-13-1) [19](#page-13-2) and [21](#page-13-3) solve for the capital stock in each sector.

**Dynamics** The optimality conditions of the problem, equation [10](#page-11-3) to [12](#page-11-0) and [17](#page-12-0) yield the conditions describing the optimal dynamics of the system around a neighborhood of the BGP. We repeat them here to ease the exposition Using an analogous expression for consumption in sector j, we can solve for the Euler equation just a function of primitives

$$
\omega_{ji}\sigma\frac{c_i(t)}{c_i(t)} = \omega_{ii}\frac{x_i(t)}{\chi_{ii}(t)}\frac{\partial F(k_i(t))}{\partial k_i(t)} - (\delta_i + g_i^k - \rho) - (1 - \omega_{ji})\left(\frac{\omega_{jj}}{\omega_{ij}}\frac{x_j(t)}{\chi_{jj}(t)}\frac{\partial F(k_j(t))}{\partial k_j(t)} - \frac{\delta_j + g_j^k - \rho}{\omega_{ij}}\right)
$$

$$
\dot{k}_i(t) = \prod_{j=1,2} \left( F(k_j(t)) - c_j(t) \right)^{\omega_{ji}} \prod_{j=1,2} \kappa_{ji}(t)^{\omega_{ji}} - (\delta_i + g_i^k) k_i(t),
$$

$$
\frac{\dot{\kappa}_{ji}(t)}{\kappa_{ji}(t)} = \frac{1}{\left(\frac{1}{1-\kappa_{ji}} - \zeta_{ji}\right)\left(\frac{1}{1-\kappa_{ij}} - \zeta_{ij}\right)} \left(\omega_{jj} \frac{x_j(t)}{\chi_{jj}(t)} \frac{\partial F(k_j(t))}{\partial k_j(t)} - \omega_{ii} \frac{x_i(t)}{\chi_{ii}(t)} \frac{\partial F(k_i(t))}{\partial k_i(t)} + \delta_i + g_i^k - \delta_j + g_j^k\right)
$$
\nwhere  $\zeta_{ji} \equiv (\omega_{ji} + \omega_{jj} \frac{\kappa_{jj}}{1-\kappa_{jj}}).$ 

$$
\dot{\kappa}_{jj}(t) = -\dot{\kappa}_{ji}(t)
$$

$$
\frac{x_i(t)}{\chi_{ii}(t)} = \prod_{j=1,2} \left( \frac{F(k_j(t)) - c_j(t)}{F(k_i(t)) - c_i(t)} \right)^{\omega_{ji}} \prod_{j=1,2} \frac{\kappa_{ji}(t)}{\kappa_{ii}(t)}^{\omega_{ji}}
$$

The Jacobian of the system computed at the steady state would characterize the speed of convergence to the steady state, which is bounded by its largest eigenvalue (in absolute value). If the spectral radius of the system is below 1 the system is stable.

## **4 Sectorial Crosswalks**

### **(a)** ISIC Rev.3



### **(b)** Mensah and de Vries (2023)



## **4.1 IPUMS: Country Specific Crosswalks**







#### **Table 6:** Switzerland



### **Table 7:** China



### **Table 8:** Costa Rica





### **Table 9:** Indonesia





### **Table 10:** India





#### **IPUMS Code IPUMS Description Aggregate Sector** 1 Agriculture, hunting, and similar services Agriculture 2 Sylviculture, forestry, similar services Agriculture 5 Fishing, aquaculture Agriculture 10 Lignite, coal, peat extraction Mining 11 Hydrocarbon extraction, and similar services Mining 13 Metallic mineral extraction, working and enrichment Mining 14 Other extraction industries Mining 15 Food industry Nondurables<br>16 Tobacco industry Nondurables<br>16 Nondurables 16 Tobacco industry 17 Textile industry **Nondurables** 18 Clothing and fur industry Nondurables 19 Leather and shoe industry Nondurables 20 Woodwork and wood products production Durables 21 Paper and carton production Nondurables 22 Publishing, printing, reproduction Nondurables 23 Coke, refinery, nuclear industry Nondurables 24 Chemical industry Nondurables 25 Rubber and plastic industry Nondurables 26 Other non-metallic mineral products manufacturing Durables 27 Metallurgy Durables 28 Metal works Durables 29 Machine and equipment manufacturing Machinery 30 Office machine and informatics materials manufacturing Electronics 31 Machine and electric appliances manufacturing Electronics<br>32 Television and radio appliances manufacturing Electronics 32 Television and radio appliances manufacturing Electronics<br>33 Medical, precision, and optic instruments manufacturing Electronics Medical, precision, and optic instruments manufacturing 34 Motor vehicle industry Transportation 35 Other transportation materials manufacturing Transportation 36 Furniture and various industries manufacturing Durables 40 Electricity, gas, heating production and distribution Services<br>41 Water collection, treatment and supply Services Water collection, treatment and supply Services 45 Construction and public works Construction<br>50 Motor vehicle sale and repair Services Motor vehicle sale and repair Services 51 Wholesale and intermediate trade Services 52 Retail sale and repair of domestic objects Services 55 Hotels and restaurants Services 60 Land transportation Services<br>
61 Water transportation Services **611 Water transportation** Services 62 Air transportation Services<br>63 Other transportation services Services Other transportation services Services 64 Postal system and telecommunications ICT 65 Finance intermediate Services 66 Insurance Services 67 Other finance and insurance activities Services<br>
70 Real estate Services 70 Real estate<br>
71 No-operator re No-operator rental ICT 72 Informatics ICT 73 Research and development ICT 74 Business-oriented services ICT Public administration Services 80 Education Services 85 Health and social work Services 90 Sewage and waste management Services 91 Activities in associations Services 92 Entertainment, cultural, sport activities Services example of the Personal services of the Services Se 95 Domestic services Services<br>
99 Extra-territorial activities Services 99 Extra-territorial activities Services Unknown

#### **Table 11:** Morocco

999 NIU (not in universe)

### **Table 12:** Mauritius



### **Table 13:** Malaysia





### **Table 14:** Nigeria









### **Table 16:** Portugal





## **Table 17:** Senegal





 $29 \frac{99}{2}$ 

### **Table 18:** Thailand



### **Table 19:** Vietnam





### **Table 20:** South Africa





